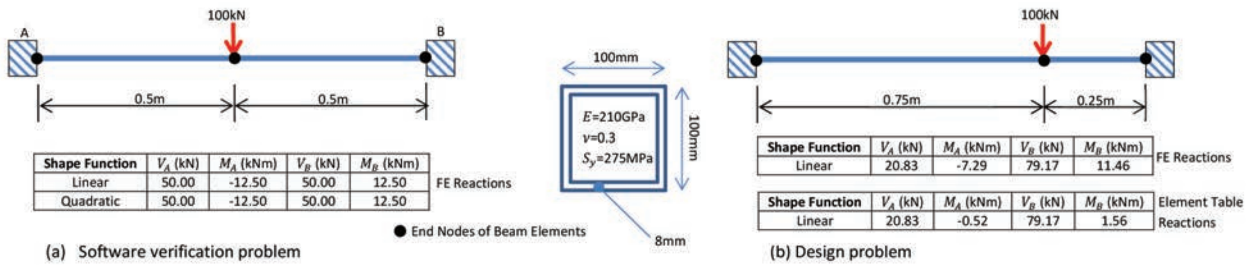
NAFEMS Benchmark Challenge

NBC07: The Solution

Modelling Beam Members with Finite Elements

Commercial finite element systems generally offer a range of beam finite elements for the engineer to model his/her beam type structure. Different finite element formulations are often provided to cope with the standard Euler-Bernoulli beam, which does not include the influence of shear deformation, and with Mindlin and/or Hybrid formulations, which do account for shear deformation as in Timoshenko's theory. The Mindlin type element, which uses the same shape functions to approximate all displacements (translations and rotations) is often offered as a variable degree element, e.g., two-noded linear shape functions or three-noded quadratic shape functions, providing the engineer with the opportunity to adopt a p-type mesh refinement strategy in addition to the usual h-type approach. The Mindlin element with both linear and quadratic shape functions was used to model the problem shown in figure 1(a) where the beam is fixed at both ends with a concentrated load at the centre of the span. A two-element mesh was used with a node at the centre of the beam where the load is applied.

The finite element reactions agree for both degrees of shape function (p=1 and p=2), and with the verification documentation supplied by the software vendor.



(a) FE Reactions

Shape Function	V_A (kN)	M_A (kNm)	V_B (kN)	M_B (kNm)
Linear	50.00	-12.50	50.00	12.50
Quadratic	50.00	-12.50	50.00	12.50

(b) FE Reactions

Shape Function	V_A (kN)	M_A (kNm)	V_B (kN)	M_B (kNm)
Linear	20.83	-7.29	79.17	11.46

(b) Element Table Reactions

Shape Function	V_A (kN)	M_A (kNm)	V_B (kN)	M_B (kNm)
Linear	20.83	-0.52	79.17	1.56

Figure 1: Beam problems and finite element reactions

On the basis of the quality of the results for the problem of figure 1(a) achieved with a minimal two–element mesh, the engineer might be led into thinking that the same quality would be achieved if the concentrated load were moved to three quarter's span as shown in figure 1(b). The finite element reactions certainly satisfy vertical and rotational equilibrium in the global sense. However, since moment equilibrium involves an unknown statically indeterminate constant moment one cannot be sure that the moment reactions are the correct values. Evidence that they might be incorrect is provided by the so– called element table results provided by the software used to produce these results, which shows moments at the supports that have been extrapolated from the integration points.

The Challenge

The reader is asked to investigate the two beam problems presented in figure 1 using the various beam elements offered in their finite element system (including and excluding shear deformation) with the aim of providing accurate results for the moment reactions, the maximum displacement and the maximum bending moment. Evidence of solution verification should be provided in terms of the convergence of the quantities of interest with mesh refinement and a prediction of the minimum number of elements required to obtain a accuracy of 1% in all quantities should be provided.

Raison d'être for the Challenge

In structural engineering the beam is a very widely used structural member. Simple beam members may be analysed by hand using strength of material solutions. However, they usually form part of a larger structure and the way in which the loads flow around the structure, as shear forces and bending moments, depends on the relative stiffness of adjacent members. Such complicated statically indeterminate structures are not generally tractable by hand methods. In such cases the finite element (FE) method is often adopted.

Often, when faced with a commercial FE system, the engineer will find that it contains a veritable plethora of beam elements each purporting to be suitable for different type of beam. Different beams will typically be based on different beam theories and different FE formulations of the particular theory. For a given beam problem, the different elements will often produce quite different results and often mesh refinement is required in order to home in on the theoretical solution.

This challenge presented two problems, the first of which was used as a software verification problem. Based on the mesh independence of the p-type refinement study conducted on the first problem, the same mesh was used for a design problem in which the load was moved from the centre to the three-quarter point. However, the results for the design problem created using the software function for tabulating element data showed moments different to those reported at the nodes. The difference observed in the results led to the challenge of determining the quantities of engineering interest, maximum deflection and moment, for the design problem.

Two Beam Theories

The primary actions seen in beams are shear and bending. For 'thin' beams, where the span to thickness ratio is large, the deformations are predominantly due to bending. However, as the span to thickness ratio decreases then deformations due to shear become significant and need to be accounted for both in terms of deflections. The Euler-Bernoulli (EB) theory ignores shear deformation and is thus appropriate to 'thin' beams whereas the Timoshenko theory includes a representation of shear deformation and is thus appropriate for thicker beams.

The kinematics of the two beam formulations differ. Both formulations assume that plane sections remain plane but whereas the EB formulation assumes that sections normal to the neutral axis in the undeformed state remain so in the deformed state, this condition is relaxed for the Timoshenko (T) formulation as illustrated in Figure 2.

Theoretical Solutions to the Design Problem

The EB solution for the design problem is available in many structural engineering texts and is presented in Figure 3.

Timoshenko solutions are less well published but may be found in such texts as *Cook et al, Concepts & Applications of Finite Element Analysis, 4th Edition, Wiley, 2002*. The theoretical solutions for the design problem using the two beam theories are shown in Table 1. In addition to the significant increase in the displacement under the load with shear deformable theory, the statics, in terms of the shear forces and moments, also change.

The theoretical displaced shapes (exaggerated by scale) for the two theories are shown in Figure 4.

Finite Element Solutions to the Design Problem

Many finite elements have been formulated based on the two beam theories already discussed. Perhaps the most common beam element based on the EB theory is the two-noded, Hermitian element which interpolates displacement as a cubic polynomial, i.e., it is capable of modelling the linear bending moments of the design problem exactly. In terms of the T theory then there is more variation in the formulation of available elements.

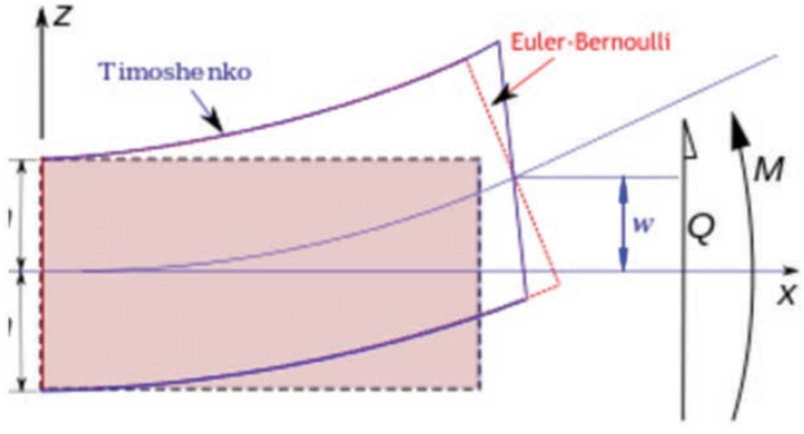


Figure 2: Euler-Bernoulli and Timoshenko kinematic assumptions - nafe.ms/2jwa57f

$V_A = V_1$ (max when $a < b$) ...... $= \frac{Pb^2}{\ell^3}(3a + b)$

$V_B = V_2$ (max when $a > b$) ...... $= \frac{Pa^2}{\ell^3}(a + 3b)$

M_A (max when $a < b$) .... $= \frac{Pab^2}{\ell^2}$

M_B (max when $a > b$) .... $= \frac{Pa^2b}{\ell^2}$

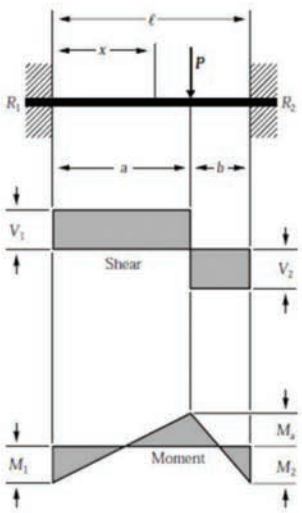
M_a (at point of load) .... $= \frac{2Pa^2b^2}{\ell^3}$

M_x (when $x < a$) .... $= R_1x - \frac{Pab^2}{\ell^2}$

$\Delta_{max}\left(\text{when } a > b \text{ at } x = \frac{2a\ell}{3a+b}\right)$ ... $= \frac{2Pa^3b^2}{3EI(3a + b)^2}$

Δ_a (at point of load) .... $= \frac{Pa^3b^3}{3EI\ell^3}$

Δ_x (when $x < a$) .... $= \frac{Pb^2x^2}{6EI\ell^3}(3a\ell - 3ax - bx)$



l=1m	V_A=15.625kN
a=0.75m	M_A=4.6875kNm
b=0.25m	V_B=84.375kN
E=210GPa	M_B=14.0625kNm
I= 4.1844E^{-6} m^4	Δ_{max}= 0.3201mm (x=0.6m)
	Δ_a=0.2501mm

Figure 3: Euler-Bernoulli solution to the Design Problem - nafe.ms/2jw1Vvu

The most widely available element is, perhaps, that based on the Mindlin formulation. This element is often of variable degree and interpolates both the displacement and rotations in the same manner, e.g. if the element is a linear element (p=1) then both displacements and rotations are interpolated linearly. The FE code used to generate the results for this response was of the Mindlin formulation with linear, quadratic and cubic degrees.

It is worth noting that a finite element system should only require a single beam element if the same element could be used reliably for both 'thick' and 'thin' beams. There is, however, a numerical issue with conforming (displacement) finite elements (CFE) formulated on T theory when the beam becomes 'thin'. This issue is known as shear-locking and it can pollute the results of 'thin' beams. For this reason many FE systems offer elements based on both beam theories and this at least allows the engineer to compare the results produced for both theories and confirm whether or not shear-locking is influencing the results.

FE results, using a two element mesh of variable degree Mindlin elements, for both the problems considered in the challenge are presented in Tables 1 and 2.

It is seen in Table 2 that whilst the shear forces and moments do not change with p-type refinement, the displacements under the load do. There is a significant change between p=1 and p=2 and only a small change, in the fifth significant digit, for p=3. The assumption, made in the challenge, that mesh independence was obtained for p=1, is clearly erroneous.

Theory	V_A (kN)	M_A (kNm)	V_B (kN)	M_B (kNm)	$M_{x=a}$ (kNm)	$\Delta_{x=a}$ (mm)
Euler-Bernoulli	15.625	-4.688	84.375	-14.063	7.031	0.250
Timoshenko	16.475	-5.112	83.525	-13.637	7.244	0.435

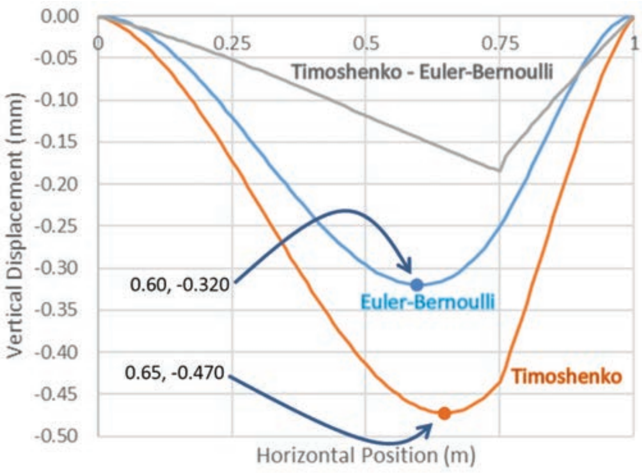
Table 1: Theoretical solutions for the Design Problem

Shape Function	V_A (kN)	M_A (kNm)	V_B (kN)	M_B (kNm)	$M_{x=a}$ (kNm)	$\Delta_{x=a}$ (mm)
Linear (p=1)	50.00	-12.50	50.00	-12.50	+12.50	0.45285
Quadratic (p=2)	50.00	-12.50	50.00	-12.50	+12.50	0.82904
Cubic (p=3)	50.00	-12.50	50.00	-12.50	+12.50	0.82908

Table 2: Convergence of quantities for the Software Verification Problem with two elements

Shape Function	V_A (kN)	M_A (kNm)	V_B (kN)	M_B (kNm)	$M_{x=a}$ (kNm)	$\Delta_{x=a}$ (mm)
Linear (p=1)	20.834	7.292	79.166	11.458	8.334	0.28551
Quadratic (p=2)	16.485	5.112	83.525	13.638	7.244	0.43487027
Cubic (p=3)	16.485	5.112	83.525	13.638	7.244	0.43487027

Table 3: Convergence of quantities for the Design Problem with two elements



The difference in the displacement between the T and EB theories is shown to illustrate the nature of the deflection due to shear deformation. It is, approximately, piecewise linear and so one might reasonably conclude that, like the EB theoretical solution, the T theory solution is also cubic in form.

Figure 4: Theoretical Displaced shapes for the Design Problem

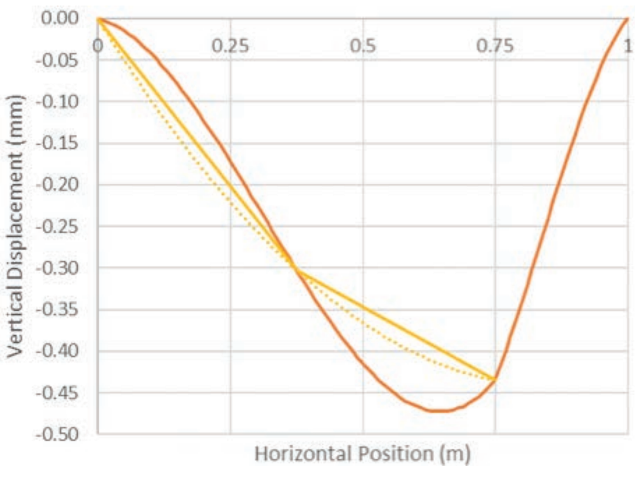
For the design problem, with the load at the three-quarter position, both shear forces, moments and displacements change as the element degree is increased between linear and quadratic. However, mesh independence does appear to be observed as the results for the cubic element are identical to those for the quadratic element.

Discussion

In the design problem the beam is moderately thick with a span to thickness ratio of 10 and the maximum deflections are 0.32 and 0.47mm respectively for the EB and T theories. Thus, if the EB displacement had been taken, the maximum deflection would have been underestimated by some 30% and this could make the difference between the beam passing and failing an SLS check on maximum deflection.

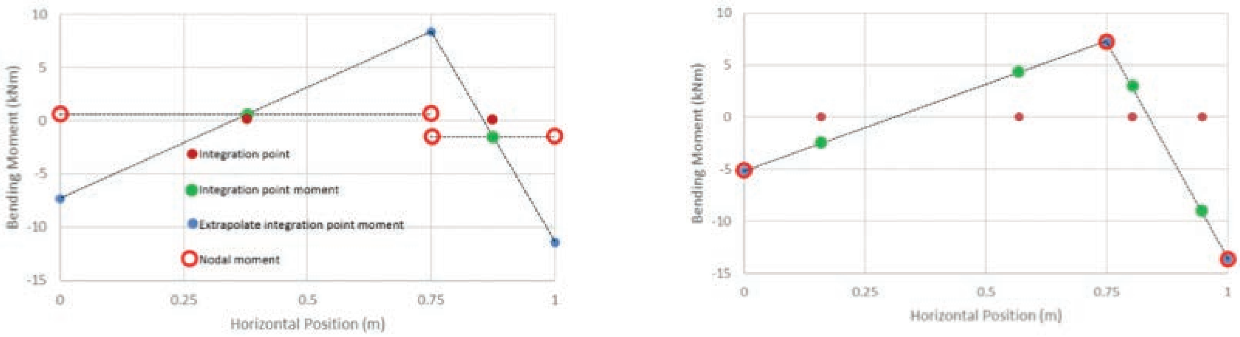
Theoretical solutions for the design problem were obtained using both beam theories. For both theories the maximum deflection was seen to occur away from the point load, at 0.60m and 0.65m respectively for the EB and T theories. Many commercial FE systems only report displacements at nodes and unless the maximum displacement occurs at the node then it will not be available to the engineer. The theoretical EB solution for the design problem can, as already noted, be recovered exactly using two cubic Hermitian beam elements. However, in order to recover the maximum displacement the engineer would have had to perform mesh refinement to 'home in' on the maximum displacement. This is an example of how poorly implemented post-processing in commercial software can frustrate the engineer's task. Had the engineer (erroneously) used an element based on EB theory and taken the maximum displacement from a two element model then he would have obtained 0.25mm which is almost 50% below the correct value! Of course, if one is adopting an inappropriate mathematical model in addition to not picking up the maximum displacement then simulation governance, the matching of numerical simulation with measured results, will be impossible.

The design problem is hyperstatic (statically indeterminate) thus the different kinematic assumptions of the two beam theories, which result in different beam stiffnesses, also, in addition to the displacements, lead to the different forces and moments. Both sets are, however, in equilibrium with the applied load. The



The exact Timoshenko displacement is shown for the design problem together with the result for the single p=2 Mindlin element. The FE displacements are, by definition, quadratic and cannot fit the exact cubic shape. The quadratic result does however appear to fit exactly at the nodes.

Figure 5: Comparison of theoretical displacements with quadratic Mindlin element



(a) Linear (p=1)

(b) Quadratic (p=2)

Figure 6: Comparison of nodal moments and those extrapolated from integration points

maximum moment for both theories is at the right hand support and it is seen that the EB theory predicts a moment of 14.06kNm whereas the T theory is about 3% less at 13.64kNm. In an Allowable Stress Design approach, this difference would lead to different factors of safety but in a ULS calculation of the plastic limit load, the collapse load would be unchanged.

For this response, a variable degree Mindlin element was used to model the Timoshenko beam theory. The advice offered by the vendor for this element is that the quadratic (p=2) element is capable of representing linearly varying bending moments exactly. The results almost bear this out except that there is a small difference in the displacement under the load for the Software Verification Problem – see Table 2 – as the degree is increased from quadratic to cubic. The quality of the result for the p=2 Mindlin element is though somewhat surprising since we know, from the theoretical solution shown in Figure 4, that the displacement field is more or less cubic. In investigating this apparent anomaly further, the displacement for a three-noded, quadratic Mindlin element was compared with the exact Timoshenko solution – see Figure 5.

The nodal displacements are, more or less, exact but clearly since they are quadratic then between nodes there is significant discrepancy. The moments inside the element are examined to see whether or not they agree with those reported at the nodes - Figure 6. In the case of the linear Mindlin element, a single integration point is used and so the variation of the internal moment field is assumed to be constant. This leads to significant differences between the internally generated moments and the nodal moments. Whilst the nodal moments are in equilibrium with the applied load, the internal moments, extrapolated from integration points, are clearly not. For the quadratic element two integration points are used and it is seen that these appear to be exact as a linear extrapolation to the nodes leads to the same values as the nodal moments.

Thus, in responding to the challenge, it might be noted that with a two-element mesh of quadratic Mindlin elements, a very close approximation to the theoretically exact solution is obtained. It is noted, however, that the maximum displacement is not available from this mesh and it has already been noted how this inadequacy might stymy the engineer's task. The same is, of course, true for moments. It is rather easy to construct a problem where the maximum moment occurs somewhere between nodes. Thus both serviceability and 'strength' calculations, based on maximum moment, might be compromised for the engineer by the inadequacies of commercial software.

In closing this response, it is important to recognise a potential cause of finite element malpractice when using Mindlin type elements. When using a linear element the internal moments (extrapolated from integration points) would not have been the same as the nodal values. In fact, in this example, they would have been significantly less than the true values - see Figure 6(a). The erroneously underestimated moments would lead, in an Allowable Stress Design, to an overestimation of the factor of safety and this is clearly of concern to the engineer.

The reason that this point is mentioned is that in some industries the standard technique for assessing structural members is based on internal stress resultants extrapolated from integration points. As demonstrated in this response, if the degree of the element is not appropriate for the loading seen by the beam then this approach can lead to significantly erroneous stress resultants that would, under code assessment, give an erroneous view of the safety of the structure.

The solution to this potentially significant issue, is to always use stress resultants calculated directly at the nodes from the basic equilibrium equations. These are guaranteed to be in equilibrium with the applied loads even if the degree of the element is inappropriate to the applied loading, and provided sufficient mesh refinement (either/both p-type and h-type) has been undertaken then these resultants will form an appropriate set on which the structure can safely be assessed. ■

The NAFEMS benchmark challenge is created and discussed by Angus Ramsay.
If you would like to discuss any of the challenge please contact **challenge@nafems.org**